

ON THE CHARACTERISTICS OF A CLASS OF GAUSSIAN PROCESSES WITHIN THE WHITE NOISE SPACE SETTING

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ABSTRACT. Using the white noise space framework, we define a class of stochastic processes which include as a particular case the fractional Brownian motion and its derivative. The covariance functions of these processes are of a special form, studied by Schoenberg, von Neumann and Krein.

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1. INTRODUCTION

Using the white noise framework, we define and study Gaussian processes whose covariance functions are the form

$$(1.1) \quad K_r(t, s) = r(t) + r(s)^* - r(t - s) - r(0), \quad t, s \in \mathbb{R},$$

where r is a continuous function such that

$$r(-t) = r(t)^*, \quad t \in \mathbb{R},$$

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and the kernel $K_r(t, s)$ is positive (in the sense of reproducing kernels) on the real line. As we will recall in the sequel, such functions r have been investigated for a long time. Still, their applications in stochastic calculus seem to have been only partially developed. We mention in particular the recent work [29, p. 103]. In that work the notion of processes with covariance measure is introduced, and stochastic processes with covariance function of the form K_r are shown to belong to this class. We note however that the methods of [29] and of the present paper are completely different.

Functions r such that the kernel $K_r(t, s)$ is positive have been characterized by von Neumann and Schoenberg when r is real; see [34, Theorem 1, p. 229]. For the case of complex-valued functions, see Krein [26] and [25], and Akhiezer, [1, pp. 267–269] and the references therein. These are functions of the form

$$(1.2) \quad r(t) = r_0 + i\gamma t - \int_{\mathbb{R}} \left\{ e^{itu} - 1 - \frac{itu}{u^2 + 1} \right\} \frac{d\sigma(u)}{u^2},$$

where $r_0 = r(0)$ and γ are real numbers with $d\sigma$ a positive measure on \mathbb{R} defined by an increasing right continuous function σ , such that

$$(1.3) \quad \int_{\mathbb{R}} \frac{d\sigma(u)}{u^2 + 1} < \infty.$$

That the form (1.2) is sufficient to insure the positivity of the kernel $K_r(t, s)$ follows from the formula

$$(1.4) \quad K_r(t, s) = \int_{\mathbb{R}} \frac{e^{itu} - 1}{u} \frac{e^{-isu} - 1}{u} d\sigma(u),$$

see for instance [27, Theorem 4 p. 115]. For a discussion of this formula, see also [32, (7) p. 25]. The idea of the proof of the converse is given in the next section.

Note that

$$K_r(t, s) = K_{r-r(0)}(t, s),$$

and therefore one can always assume that $r(0) = 0$.

In the real valued case, with $r(0) = 0$, the function r takes the form

$$r(t) = \int_{\mathbb{R}} \frac{1 - \cos(tu)}{u^2} d\sigma(u).$$

The fractional Brownian motion then corresponds to the choice

$$d\sigma(u) = \frac{1}{2\pi} |u|^{1-2H} du, \quad H \in (0, 1),$$

giving

$$(1.5) \quad r(t) = \frac{V_H}{2}|t|^{2H}, \quad \text{with} \quad V_H = \frac{\Gamma(2-2H)\cos(\pi H)}{\pi(1-2H)H},$$

and

$$K_r(t, s) \stackrel{\text{def}}{=} k_H(t, s) = \frac{V_H}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}),$$

with Γ denoting the Gamma function, as can be seen using the formulas

$$\begin{aligned} \int_{\mathbb{R}} \frac{1 - \cos(tu)}{u^{2H+1}} du &= -2|t|^{2H} \cos(\pi H) \Gamma(-2H) \\ \int_{\mathbb{R}} \frac{1 - \cos(tu)}{u^2} du &= \pi|t|. \end{aligned}$$

When furthermore $H = 1/2$, then $V_H = 1$, $r(t) = |t|/2$ and for $t, s \geq 0$, $K_r(t, s) = \min(t, s)$.

By a theorem of Kolmogorov there exists a Gaussian stochastic process $\{B_H(t)\}$ indexed by \mathbb{R} , which is called the fractional Brownian motion (with Hurst parameter $H \in (0, 1)$), such that

$$k_H(t, s) = E(B_H(t)B_H(s)), \quad t, s \in \mathbb{R}.$$

Stochastic calculus for B_H has been developed for quite some time; see for instance [12], [15], [18], [6]. In a subsequent paper we show how most of the results of these works are extended to the case of general covariance functions of the form (1.1) above.

We mention also that functions r of the form (1.2) appear first in the work of Paul Lévy [31], in the result characterizing characteristic functions of infinitely divisible laws. More precisely, the characteristic function of a random variable is infinitely divisible if and only if, it is of the form $\exp r(t)$, where $r(t)$ is of form (1.2); see [33, Representation Theorem]. Similarly, $(Z_u)_{u \geq 0}$ is an infinitely divisible random process if and only if

$$E(e^{itZ_u}) = e^{-ur(t)},$$

where r is of the form (1.2), with $r(0) = 0, \gamma = 0$, and is called the *characteristic exponent of the Lévy process*. See [7, p. 12], [31, formula (9), p. 353]. This is the Lévy-Khintchine formula, see [31, (formula (9) p. 353)]. We also mention that positive kernels of the form K_r appear in the theory of Dirichlet spaces; see [13, p. 5-12]. These aspects of the theory of kernels of the form K_r will not be studied in the present paper.

The paper consists of 7 sections including the introduction, and its outline is as follows. In Section 2 we study and characterize the reproducing kernel Hilbert space associated with a kernel $K_r(t, s)$. In Section 3 we associate with certain kernels $K_r(t, s)$ an operator which will play a key role in the construction of the stochastic process with covariance $K_r(t, s)$. This paper uses Hida's white noise space theory, and in Section 4 we review the main features from white noise space theory which we will subsequently use. In Section 5 we recall the definition of the Wick product and of the Kondratiev space. Stochastic processes with covariance function $K_r(t, s)$ are built in Section 6, and their derivatives are studied in Section 7.

Some of the results presented here have been announced in the note [4]. The results of the present work are to be used in a subsequent paper where stochastic analysis of the processes considered here is developed.

Finally a word on notation. We denote the Fourier transform by

$$(1.6) \quad \hat{f}(u) = \int_{\mathbb{R}} e^{-iux} f(x) dx.$$

The inverse transform is then given by

$$(1.7) \quad \check{f}(u) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{iux} f(x) dx.$$

The same notations are used for the Fourier transform and inverse Fourier transform of distributions.

We set

$$\mathbb{N} = \{1, 2, 3, \dots\} \quad \text{and} \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

and denote by ℓ the set of sequences

$$(1.8) \quad (\alpha_1, \alpha_2, \dots),$$

indexed by \mathbb{N} with values in \mathbb{N}_0 , for which only a finite number of elements $\alpha_j \neq 0$.

2. SOME REMARKS ON THE KERNELS $K_r(t, s)$

As already mentioned, real-valued functions r for which the kernel $K_r(t, s)$ is positive on the real line (in the sense of reproducing kernels) were characterized by Schoenberg and von Neumann. For complex-valued functions, and by different methods, the following theorem has been given by Krein in 1944; see [27, Theorem 2, p. 256]. See also [1,

§9 p. 268]. In Krein's result, the case $a = \infty$ is allowed, and then, $t, s \in \mathbb{R}$.

Theorem 2.1. [27, Theorem 2, p. 256] *The kernel $K_r(t, s) = r(t) + r(s)^* - r(t - s) - r(0)$ is positive for $t, s \in [-a, a]$ if and only if it is of the form (1.2):*

$$r(t) = r(0) + i\gamma t - \int_{\mathbb{R}} \left\{ e^{itu} - 1 - \frac{itu}{u^2 + 1} \right\} \frac{d\sigma(u)}{u^2}, \quad t \in [-a, a].$$

For completeness, let us recall that Akhiezer's proof [1, pp. 268-270] goes along the following lines; one first shows that r satisfies an inequality of the form

$$(2.1) \quad |r(t)| \leq M(1 + |t|^3)$$

for some positive number M (we recall the proof of this inequality in the sequel; see Lemma 2.4 below). One then shows that the function $H(z) = z^2 \int_0^\infty r(t)^* e^{itz} dt$, which, in view of (2.1), is analytic in the open upper half-plane, satisfies (see [1, (2) pp. 268] and [28, p. 227])

$$\frac{H(z) - H(w)^*}{z - w^*} = zw^* \iint_{\mathbb{R}_+^2} K_r(t, s) e^{itz} e^{-isw^*} dt ds,$$

and in particular has a positive imaginary part in the open upper half-plane. To conclude the proof, one uses Herglotz's representation formula for analytic functions with a positive imaginary part in the open upper half-plane (see for instance [11, Theorem 4.7, p. 25], [2, Theorem 2 p. 220]).

Formula (1.4) allows to characterize the reproducing kernel Hilbert space associated with K_r in terms of de Branges spaces. See Theorem 2.3 below. We first make the following remarks: Let χ_t denote the function of a real variable u

$$\chi_t(u) = \frac{e^{itu} - 1}{iu}, \quad t \in \mathbb{R},$$

and let \mathcal{M}_T denote the closed linear span in $\mathbf{L}_2(d\sigma)$ of the χ_t for $|t| \leq T$. Assume that $\mathcal{M}_T \neq \mathbf{L}_2(d\sigma)$. Then, \mathcal{M}_T is a reproducing kernel Hilbert space with reproducing kernel of the form

$$\frac{A(T, \lambda)A(T, \omega)^* - B(T, \lambda)B(T, \omega)^*}{-i(\lambda - \omega^*)}$$

where $A(T, \lambda), B(T, \lambda)$ are entire functions of finite exponential type. See [10], [17].

Remark 2.2. The spaces \mathcal{M}_T were introduced by de Branges and play a key role in prediction theory. See [10, 17, 16]. When $d\sigma(u) = du$ we have

$$\mathcal{M}_T = \mathbf{H}_2 \ominus e_T \mathbf{H}_2,$$

where \mathbf{H}_2 denotes the Hardy space of the open half-plane, and where $e_T(z) = e^{izT}$, $B(T, \cdot) = 1$ and $A(T, \cdot) = \chi_T$, with the reproducing kernel given by

$$\frac{1 - e_T(\lambda)e_T(\omega^*)^*}{-i(\lambda - \omega^*)}.$$

Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space of rapidly decreasing functions, and let $\mathcal{S}'(\mathbb{R})$ denote the topological dual of $\mathcal{S}(\mathbb{R})$, that is, the space of tempered distributions. In view of the next two results, we recall the following: Condition (1.3) insures that the measure $d\sigma$ has a Fourier transform $\widehat{d\sigma}$ which is a tempered distribution. Furthermore, this Fourier transform induces a distribution $\widehat{d\sigma}(t-s)$ on the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ of functions of two real variables via the formula

$$(2.2) \quad \langle \widehat{d\sigma}(t-s), \phi(t, s) \rangle = \int_{\mathbb{R}} \left(\iint_{\mathbb{R}^2} e^{-i(t-s)u} \phi(t, s) dt ds \right) d\sigma(u).$$

When $\int_{\mathbb{R}} d\sigma(u) < \infty$, we have that

$$\begin{aligned} \langle \widehat{d\sigma}(t-s), \phi(t, s) \rangle &= \iint_{\mathbb{R}^2} \left\{ \int_{\mathbb{R}} e^{-i(t-s)u} d\sigma(u) \right\} \phi(t, s) dt ds \\ &= \iint_{\mathbb{R}^2} \widehat{d\sigma}(t-s) \phi(t, s) dt ds. \end{aligned}$$

Theorem 2.3. Let $T < \infty$. The reproducing kernel Hilbert space $\mathcal{H}_T(K_r)$ associated with $K_r(t, s)$ for $t, s \in [-T, T]$ consists of functions of the form

$$(2.3) \quad F(t) = \int_{\mathbb{R}} \frac{e^{itu} - 1}{iu} f(u) d\sigma(u), \quad t \in [-T, T], \quad f \in \mathcal{M}_T$$

with norm

$$(2.4) \quad \|F\|_{\mathcal{H}_T(K_r)} = \|f\|_{\mathbf{L}_2(d\sigma)}.$$

Moreover, by extending $F(t)$ to the real line by formula (2.3), F defines a tempered distribution and

$$F'(t) = 2\pi \overset{\vee}{f} d\sigma(t)$$

in the sense of distributions, and where $\overset{\vee}{f} d\sigma$ denotes the inverse Fourier transform of the tempered distribution defined by $f d\sigma$.

Proof: Let F be of the form (2.3) and assume $F \equiv 0$. Then f is orthogonal to all χ_t , $t \in [-T, T]$ and therefore $f = 0$ since $f \in \mathcal{M}_T$. Thus, (2.4) defines a norm. Furthermore, the choice $f = (\chi_s)^*$ in (2.3) leads to

$$\begin{aligned} \langle F, K_r(\cdot, s) \rangle_{\mathcal{H}_T(K_r)} &= \langle f, (\chi_s)^* \rangle_{\mathbf{L}_2(d\sigma)} \\ &= \int_{\mathbb{R}} f(u) \chi_s(u) d\sigma(u) = F(s), \end{aligned}$$

which proves the first claim.

We note that the function $F(t)$ extends to a continuous function on \mathbb{R} . Indeed, for every $t, h \in \mathbb{R}$ we have

$$(2.5) \quad \left| \frac{e^{i(t+h)u} - e^{itu}}{iu} \right| = \left| \frac{e^{ihu} - 1}{iu} \right| \leq \begin{cases} |h| & \text{if } |u| \leq 1, \\ \frac{2}{|u|} & \text{if } |u| > 1. \end{cases}$$

Using these inequalities we use the dominated convergence theorem to prove that

$$\lim_{h \rightarrow 0} F(t+h) = F(t), \quad t \in \mathbb{R}.$$

Furthermore, (2.5) leads to the bound:

$$|F(t)| \leq |t| \int_{-1}^1 |f(u)| d\sigma(u) + 2 \int_{|u| \geq 1} \left| \frac{f(u)}{u} \right| d\sigma(u).$$

We recall that σ is assumed right continuous. When it has a jump at 0, we define

$$F(t) = t f(0)(\sigma(0) - \sigma(0_-)) + \int_{\mathbb{R}} f(u) \chi_s(u) d\sigma_1(u),$$

where $d\sigma_1$ has no jump at 0. The function F is in particular slowly growing, and therefore defines a tempered distribution (see [36, Théorème VI p. 239], [5, §4, p. 110]).

Let $\varphi \in \mathcal{S}(\mathbb{R})$. The integral

$$\int_{\mathbb{R}} f(u) \varphi(u) d\sigma(u) = \int_{\mathbb{R}} \frac{f(u)}{u+i} ((u+i)\varphi(u)) d\sigma(u)$$

exists, since $(u+i)\varphi(u)$ is bounded and since $1/(u+i) \in \mathbf{L}_2(d\sigma)$. Thus,

$$\begin{aligned} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| \frac{e^{itu} - 1}{iu} f(u) \varphi'(t) \right| d\sigma(u) \right) dt &\leq \int_{|u| \leq 1} |f(u)| d\sigma(u) \int_{\mathbb{R}} |t \varphi'(t)| dt + \\ &+ 2 \int_{|u| > 1} \left| \frac{f(u)}{u} \right| d\sigma(u) \int_{\mathbb{R}} |\varphi'(t)| dt < \infty. \end{aligned}$$

Using Fubini's theorem, we have

$$\int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} \frac{e^{itu} - 1}{iu} f(u) d\sigma(u) \right\} \varphi'(t) dt = \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} \frac{e^{itu} - 1}{iu} \varphi'(t) dt \right\} f(u) d\sigma(u),$$

and by integration by parts, we obtain

$$\begin{aligned} \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} \frac{e^{itu} - 1}{iu} \varphi'(t) dt \right\} f(u) d\sigma(u) &= - \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} e^{itu} \varphi(t) dt \right\} f(u) d\sigma(u) \\ &= -2\pi \int_{\mathbb{R}} \overset{\vee}{\varphi}(u) f(u) d\sigma(u). \end{aligned}$$

Thus,

$$\int_{\mathbb{R}} F(t) \varphi'(t) dt = -2\pi \int_{\mathbb{R}} \overset{\vee}{\varphi}(u) f(u) d\sigma(u),$$

and therefore we obtain on the one hand

$$\langle F, \varphi' \rangle = -2\pi \left\langle f d\sigma, \overset{\vee}{\varphi} \right\rangle = -2\pi \left\langle f d\sigma, \varphi \right\rangle.$$

On the other hand,

$$\langle F, \varphi' \rangle = - \langle F', \varphi \rangle$$

Thus, $F' = 2\pi \overset{\vee}{f} d\sigma$. □

In preparation for the proof of Theorem 2.5 below we now prove inequality (2.1).

Lemma 2.4. *Assume the kernel $K_r(t, s)$ to be positive in \mathbb{R} . Then (2.1) is in force, that is*

$$|r(t)| \leq M(1 + |t|^3)$$

for some positive number M .

Proof: We follow the arguments in [1, pp. 264-265], with slight modifications. We first note that we may assume that $r(0) = 0$. The positivity of the kernel $K_r(t, s)$ implies that the matrix

$$\begin{pmatrix} K_r(t, t) & K_r(t, -t) \\ K_r(-t, t) & K_r(-t, -t) \end{pmatrix}$$

has a non-negative determinant. Therefore,

$$|2r(t) - r(2t)| \leq |2\operatorname{Re} r(t)|,$$

and thus

$$(2.6) \quad |r(2t)| \leq 4|r(t)|.$$

Let

$$R(t) = \frac{|r(t)|}{1 + |t|^3}.$$

Then, (2.6) implies that

$$(2.7) \quad R(2t) \leq \frac{4(1 + |t|^3)}{1 + 8|t|^3} R(t).$$

Let $T_0 \in \mathbb{R}_+$ be such that:

$$|t| \geq T_0 \implies \frac{4(1 + |t|^3)}{1 + 8|t|^3} \leq 1.$$

It follows from (2.7) that $R(t)$ is bounded in \mathbb{R} by an expression of the form $M(1 + |t|^3)$ for some $M > 0$. In fact, since r is continuous, one may take

$$M = \max_{t \in [0, T_0]} |r(t)|.$$

□

Theorem 2.5. *It holds that*

$$\frac{\partial^2}{\partial t \partial s} K_r(t, s) = r''(t - s) = \widehat{d\sigma}(s - t)$$

*in the sense of distributions. Furthermore, for $\varphi \in \mathcal{S}(\mathbb{R})$, $\overset{\vee}{d\sigma} * \varphi$ is a function and it holds that $\varphi \in \mathcal{S}(\mathbb{R})$*

$$(2.8) \quad \langle d\sigma \widehat{\varphi}, (\widehat{\varphi})^* \rangle_{\mathcal{S}'(\mathbb{R}), \mathcal{S}(\mathbb{R})} = \int_{\mathbb{R}} \varphi(-u)^* (\overset{\vee}{d\sigma} * \varphi)(u) du$$

Before the proof, we make the following observation. We note that the right hand side of (2.8) can formally be rewritten as

$$\int_{\mathbb{R}} \varphi(-u)^* \left(\int_{\mathbb{R}} \overset{\vee}{d\sigma}(u - v) \varphi(v) dv \right) du$$

where in general

$$\int_{\mathbb{R}} \overset{\vee}{d\sigma}(u - v) \varphi(v) dv$$

is not a real integral, but an abuse of notation.

In the proof of the theorem, use is made of properties of the space \mathcal{O}_M of multiplication operator in $\mathcal{S}(\mathbb{R})$ (also called C^∞ functions slowly decreasing at infinity), see [37, p. 275], and of the space \mathcal{O}'_C of distributions rapidly decreasing at infinity, see [37, p. 315]. Recall ([37, Theorem 30.3, p. 318]) that the Fourier transform is one-to-one from \mathcal{O}_M onto \mathcal{O}'_C and from \mathcal{O}'_C onto \mathcal{O}_M .

Proof of Theorem 2.5: Since $d\sigma$ defines a distribution in $\mathcal{S}'(\mathbb{R})$, we have that $\overset{\vee}{d}\sigma \in \mathcal{S}'(\mathbb{R})$; see for instance [37, Theorem 25.6, p. 276]. Let $\varphi \in \mathcal{S}(\mathbb{R})$. The convolution $\overset{\vee}{d}\sigma * \varphi$ is a function, and belongs to \mathcal{O}_M ; see [36, p.248]. So, by [37, Theorem 30.3, p. 318]

$$(2.9) \quad \widehat{\overset{\vee}{d}\sigma * \varphi} \in \mathcal{O}'_C.$$

Now we compute (2.9) using [37, Theorem 30.4 p. 319] with (in the notation of that book) $S = \overset{\vee}{d}\sigma \in \mathcal{S}'(\mathbb{R})$ and $T = \varphi \in \mathcal{S}(\mathbb{R}) \subset \mathcal{O}'_C$ (see [37, Example 30.1, p. 315] for the latter inclusion), and obtain

$$\widehat{\overset{\vee}{d}\sigma * \varphi} = d\sigma \widehat{\varphi},$$

which is a measure. Using the fact that $\mathcal{O}'_C \subset \mathcal{S}'(\mathbb{R})$ (see [37, p. 318], [36]), we have

$$(2.10) \quad \langle d\sigma \widehat{\varphi}, \widehat{\varphi}^* \rangle_{\mathcal{S}'(\mathbb{R}), \mathcal{S}(\mathbb{R})} = \int_{\mathbb{R}} |\widehat{\varphi}|^2 d\sigma.$$

But for $\psi \in \mathcal{S}'(\mathbb{R})$ and $\varphi \in \mathcal{S}(\mathbb{R})$ we have:

$$\left\langle \widehat{\psi}, \varphi \right\rangle_{\mathcal{S}'(\mathbb{R}), \mathcal{S}(\mathbb{R})} = \langle \psi, \widehat{\varphi} \rangle_{\mathcal{S}'(\mathbb{R}), \mathcal{S}(\mathbb{R})}.$$

Let $\phi \in \mathcal{S}(\mathbb{R})$. Then the function

$$\psi \quad : \quad u \mapsto \varphi(-u)^*$$

is also in $\mathcal{S}(\mathbb{R})$ and we have:

$$\begin{aligned} \int_{\mathbb{R}} \phi(-u)^* (\overset{\vee}{d}\sigma * \varphi)(u) du &= \left\langle \overset{\vee}{d}\sigma * \varphi, \psi \right\rangle_{\mathcal{S}'(\mathbb{R}), \mathcal{S}(\mathbb{R})} \\ &= \left\langle d\sigma \widehat{\varphi}, \widehat{\psi} \right\rangle_{\mathcal{S}'(\mathbb{R}), \mathcal{S}(\mathbb{R})} \\ &= \langle d\sigma \widehat{\varphi}, \widehat{\varphi}^* \rangle_{\mathcal{S}'(\mathbb{R}), \mathcal{S}(\mathbb{R})}, \end{aligned}$$

since $\widehat{\psi} = (\widehat{\varphi})^*$. Thus, (2.10) can be rewritten as:

$$\langle d\sigma \widehat{\varphi}, \widehat{\varphi}^* \rangle_{\mathcal{S}'(\mathbb{R}), \mathcal{S}(\mathbb{R})} = \int_{\mathbb{R}} \varphi(-u)^* (d\sigma \overset{\vee}{*} \varphi)(u) du.$$

Now let $\phi \in \mathcal{S}(\mathbb{R}^2)$: We have

$$\begin{aligned} \left\langle \frac{\partial^2}{\partial t \partial s} K_r, \phi \right\rangle &= \left\langle K_r, \frac{\partial^2}{\partial t \partial s} \phi \right\rangle \\ &= \iint_{\mathbb{R}^2} K_r(t, s) \frac{\partial^2}{\partial t \partial s} \phi(t, s) dt ds \\ &= \iint_{\mathbb{R}^2} r(t) \frac{\partial^2}{\partial t \partial s} \phi(t, s) dt ds \\ &\quad + \iint_{\mathbb{R}^2} r(s)^* \frac{\partial^2}{\partial t \partial s} \phi(t, s) dt ds \\ &\quad - \iint_{\mathbb{R}^2} r(t-s) \frac{\partial^2}{\partial t \partial s} \phi(t, s) dt ds. \end{aligned}$$

Since r satisfies inequality (2.1), the above integrals make sense; the first and the second integrals on the right handside are identically zero since ϕ is a Schwartz function. Thus, since

$$\begin{aligned} \langle r''(t-s), \phi(t, s) \rangle &= - \left\langle r(t-s), \frac{\partial^2}{\partial t \partial s} \phi \right\rangle \\ &= - \iint_{\mathbb{R}^2} r(t-s) \frac{\partial^2}{\partial t \partial s} \phi(t, s) dt ds, \end{aligned}$$

it follows that

$$\left\langle \frac{\partial^2}{\partial t \partial s} K_r, \phi \right\rangle = \langle r''(t-s), \phi \rangle.$$

Using (1.2)

$$r(t-s) = i\gamma(t-s) - \int_{\mathbb{R}} \left\{ e^{i(t-s)u} - 1 - \frac{i(t-s)u}{u^2+1} \right\} \frac{d\sigma(u)}{u^2},$$

we get

$$\begin{aligned} \iint_{\mathbb{R}^2} r(t-s) \frac{\partial^2}{\partial t \partial s} \phi(t, s) dt ds &= \iint_{\mathbb{R}^2} i\gamma(t-s) \frac{\partial^2}{\partial t \partial s} \phi(t, s) dt ds - \\ &\quad - \iint_{\mathbb{R}^2} \left\{ \int_{\mathbb{R}} \left\{ e^{i(t-s)u} - 1 - \frac{i(t-s)u}{u^2+1} \right\} \frac{d\sigma(u)}{u^2} \right\} \frac{\partial^2}{\partial t \partial s} \phi(t, s) dt ds. \end{aligned}$$

The first integral on the right hand-side vanishes since ϕ is a Schwartz function. The function

$$K(t, s, u) = \begin{cases} \left\{ e^{i(t-s)} - 1 - \frac{i(t-s)u}{u^2+1} \right\} \frac{1}{u^2}, & \text{if } u \neq 0, \\ -\frac{(t-s)^2}{2}, & \text{if } u = 0, \end{cases}$$

is continuous since

$$\lim_{u \rightarrow 0} \left\{ e^{i(t-s)} - 1 - \frac{i(t-s)u}{u^2+1} \right\} \frac{1}{u^2} = -\frac{(t-s)^2}{2}.$$

Moreover, we have the bounds

$$|K(t, s, u)| \leq \begin{cases} \frac{(t-s)^2 + |t-s|}{u^2+1} & \text{if } |u| < 1, \\ \frac{4+|t-s|}{u^2+1} & \text{if } |u| \geq 1. \end{cases}$$

Therefore,

$$\begin{aligned} & \iint_{\mathbb{R}^2} \int_{\mathbb{R}} \left| K(t, s, u) \frac{\partial^2}{\partial t \partial s} \phi(t, s) \right| d\sigma(u) dt ds \\ & \leq \iint_{\mathbb{R}^2} \left| \frac{\partial^2}{\partial t \partial s} \phi(t, s) \right| \left\{ \int_{\mathbb{R}} |K(t, s, u)| d\sigma(u) \right\} dt ds \\ & \leq \iint_{\mathbb{R}^2} \left| \frac{\partial^2}{\partial t \partial s} \phi(t, s) \right| \left\{ \int_{|u| < 1} \frac{(t-s)^2 + |t-s|}{u^2+1} d\sigma(u) \right\} dt ds \\ & \quad + \iint_{\mathbb{R}^2} \left| \frac{\partial^2}{\partial t \partial s} \phi(t, s) \right| \left\{ \int_{|u| \geq 1} \frac{4+|t-s|}{u^2+1} d\sigma(u) \right\} dt ds \\ & < K \iint_{\mathbb{R}^2} \left| \frac{\partial^2}{\partial t \partial s} \phi(t, s) (|t-s| + 2)^2 \right| dt ds < \infty \end{aligned}$$

where

$$K = \int_{\mathbb{R}} \frac{d\sigma(u)}{u^2+1} < \infty.$$

By Fubini's theorem and integration by parts we obtain

$$\begin{aligned}
& \int_{\mathbb{R}^2} \left\{ \int_{\mathbb{R}} \left\{ e^{i(t-s)u} - 1 - \frac{i(t-s)u}{u^2+1} \right\} \frac{d\sigma(u)}{u^2} \right\} \frac{\partial^2}{\partial t \partial s} \phi(t, s) dt ds = \\
&= \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}^2} \left\{ e^{i(t-s)u} - 1 - \frac{i(t-s)u}{u^2+1} \right\} \frac{1}{u^2} \frac{\partial^2}{\partial t \partial s} \phi(t, s) dt ds \right\} d\sigma(u) \\
&= \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}^2} e^{i(t-s)u} \phi(t, s) dt ds \right\} d\sigma(u) \\
&= \int_{\mathbb{R}^2} \left\{ \int_{\mathbb{R}} e^{-i(s-t)u} d\sigma(u) \right\} \phi(t, s) dt ds,
\end{aligned}$$

and by (2.2) we conclude that

$$\langle r''(t-s), \phi(t, s) \rangle = \langle \widehat{d\sigma}(s-t), \phi(t, s) \rangle.$$

Thus,

$$\frac{\partial^2}{\partial t \partial s} K_r(t, s) = r''(t-s) = \widehat{d\sigma}(s-t).$$

□

3. THE OPERATOR T_m

We now focus on the case $d\sigma(u) = m(u)du$ in (1.2), where m is a positive and measurable function such that

$$(3.1) \quad \int_{\mathbb{R}} \frac{m(u)du}{u^2+1} < \infty.$$

We define an (unbounded in general) operator T_m by

$$(3.2) \quad \widehat{T_m f}(u) \stackrel{\text{def.}}{=} \sqrt{m(u)} \widehat{f}(u),$$

where \widehat{f} denotes the Fourier transform of f ; see (1.6). The domain of T_m ,

$$(3.3) \quad \text{dom}(T_m) \stackrel{\text{def.}}{=} \left\{ f \in \mathbf{L}_2(\mathbb{R}) : \int_{\mathbb{R}} m(u) |\widehat{f}(u)|^2 du < \infty \right\},$$

contains in particular the Schwartz space $\mathcal{S}(\mathbb{R})$ since m satisfies (3.1) and since the Fourier transform maps $\mathcal{S}(\mathbb{R})$ into itself.

When m is summable, the integral in (3.3) can be rewritten as a double integral as explained in the previous section:

$$(3.4) \quad \int_{\mathbb{R}} m(u) |\widehat{f}(u)|^2 du = \iint_{\mathbb{R}^2} f(t) f(s)^* \widehat{m}(t-s) dt ds.$$

When

$$(3.5) \quad m(u) = \frac{1}{2\pi} |u|^{1-2H},$$

the operator T_m reduces, up to a multiplicative constant, to the operator M_H defined in [19, (2.10) p. 304] and in [8, Definition 3.1 p. 354], and the function $r(t)$ in (1.2) is given by (1.5). We note that the set (3.3) has been introduced in [35, Theorem 3.1, p. 258] for m of the form (3.5). Multiplying (3.5) by

$$\frac{2\pi H(1-2H)}{\Gamma(2-2H) \cos(\pi H)},$$

that is, considering

$$m(u) = \frac{H(1-2H)}{\Gamma(2-2H) \cos(\pi H)} |u|^{1-2H}$$

and using [20, Formula 12, p. 170], that is, in the sense of distributions,

$$\int_{\mathbb{R}} |x|^\lambda e^{iux} dx = -2\Gamma(1+\lambda) \sin\left(\frac{\pi\lambda}{2}\right) |u|^{-\lambda-1}, \quad \lambda \notin \mathbb{Z},$$

leads to (with $\lambda = 1-2H$)

$$\begin{aligned} \widehat{m}(u) &= -\frac{H(1-2H)\Gamma(2-2H) \sin\left(\frac{\pi(1-2H)}{2}\right)}{\Gamma(2-2H) \cos(\pi H)} |u|^{2H-2} \\ &= 2H(2H-1) |u|^{2H-2} \end{aligned}$$

See [15, (2.1), p. 584]. The norm $|f|_\phi^2$ defined in [15, (2.2), p. 584] is then equal to (3.4).

The operator T_m will play a key role in the sequel. We now study its main properties.

Lemma 3.1. *For every $t \in \mathbb{R}$, the function*

$$I_t \stackrel{\text{def.}}{=} \begin{cases} 1_{[0,t]}, & \text{if } t > 0, \\ 1_{[t,0]}, & \text{if } t < 0, \end{cases}$$

belongs to the domain of T_m .

Proof: We consider the case $t > 0$. The case where $t < 0$ is treated in a similar way. We have

$$\begin{aligned} \int_{\mathbb{R}} m(u) |\widehat{1_{[0,t]}}(u)|^2 du &= \int_{-\infty}^{-1} m(u) \left| \frac{e^{-itu} - 1}{-iu} \right|^2 du + \\ &+ \int_{-1}^1 m(u) \left| \frac{e^{-itu} - 1}{-iu} \right|^2 du + \\ &+ \int_1^{\infty} m(u) \left| \frac{e^{-itu} - 1}{-iu} \right|^2 du. \end{aligned}$$

The first and last integrals converge in view of (3.1), and the second is trivially convergent. \square

Theorem 3.2. *The operator T_m is self-adjoint and closed. It is bounded if and only if m is bounded.*

Proof: For f and g in the domain of T_m we have

$$\langle f, T_m g \rangle_{\mathbf{L}_2(\mathbb{R})} = \langle T_m f, g \rangle_{\mathbf{L}_2(\mathbb{R})}.$$

Thus, $T_m \subset T_m^*$ and the operator T_m is hermitian. We show that it is self-adjoint: let $g \in \text{dom}(T_m^*)$. The map $f \rightarrow \langle T_m f, g \rangle_{\mathbf{L}_2(\mathbb{R})}$ is continuous and so is the map

$$f \rightarrow \langle \widehat{T_m f}, \widehat{g} \rangle_{\mathbf{L}_2(\mathbb{R})} = \langle \sqrt{m} \widehat{f}, \widehat{g} \rangle_{\mathbf{L}_2(\mathbb{R})},$$

and the map

$$\widehat{f} \rightarrow \langle \widehat{f}, \sqrt{m} \widehat{g} \rangle_{\mathbf{L}_2(\mathbb{R})}$$

is also continuous. By Riesz representation theorem, $\sqrt{m} \widehat{g} \in \mathbf{L}_2(\mathbb{R})$, hence $g \in \text{dom}(T_m)$ and we get $T_m^* \subset T_m$.

We now show that T_m is closed: let $f_n \rightarrow f$ and $T_m f_n \rightarrow g$. We have $\widehat{f_n} \rightarrow \widehat{f}$. Thus $T_m f_n \rightarrow g$ leads to $\widehat{T_m f_n} \rightarrow \widehat{g}$, and thus $\sqrt{m} \widehat{f_n} \rightarrow \widehat{g}$. By [14, Théorème 2.3, p. 95] there exists a subsequence n_k such that $\widehat{f_{n_k}} \rightarrow \widehat{f}$ point-wise a.e. and so $\widehat{T_m f_{n_k}} \rightarrow \widehat{g}$ point-wise a.e., and we have $\widehat{T_m f} = \widehat{g}$, a.e.

Finally we show that the operator T_m is bounded if and only if m is bounded. First, if m is bounded then there exists a $K > 0$ such that $|m(u)| < K$ for any $u \in \mathbb{R}$ and we get, for any $f \in \mathbf{L}_2(\mathbb{R})$

$$\int_{\mathbb{R}} m(u) |\widehat{f}(u)|^2 du < K \int_{\mathbb{R}} |\widehat{f}(u)|^2 du,$$

so T_m is bounded since the Fourier transform is an isometry. Now if T_m is bounded, then there exists a $K \in \mathbb{R}$ such that for any $f \in \mathbf{L}_2(\mathbb{R})$

$$\int_{\mathbb{R}} m(u) \left| \widehat{f}(u) \right|^2 du \leq K \int_{\mathbb{R}} \left| \widehat{f}(u) \right|^2 du.$$

Assume that m is unbounded. Then for any $N \in \mathbb{N}$ there exists a measurable set E_N such that $\lambda(E_N) > 0$ (λ denotes the Lebesgue measure) and $m(u) \geq N$ on E_N , where, without loss of generality, one may take E_N such that $\lambda(E_N) \leq 1$. Define f_n such that $\widehat{f}_n = 1_{E_N}$ on E_N . We then have

$$N \int_{E_N} \left| \widehat{f}_n(u) \right|^2 du \leq \int_{E_N} m(u) \left| \widehat{f}_n(u) \right|^2 du \leq K \int_{E_N} \left| \widehat{f}_n(u) \right|^2 du,$$

hence $N \leq K$, but this is impossible, so m is bounded. \square

For $m(u) = \frac{1}{2\pi} |u|^{1-2H}$, we have that

$$(3.6) \quad \text{supp } T_m I_t \subset \text{supp } I_t, \quad t \in \mathbb{R}.$$

In general this property will not hold, as we now illustrate with a counterexample. This example is of particular importance to mark the difference between our approach and the approach presented in [3].

Example 3.3. Let $m(u) = u^4 e^{-2u^2}$. We have

$$\begin{aligned} (T_m 1_{[0,t]})(s) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{isu} u^2 e^{-u^2} \cdot \frac{e^{-itu} - 1}{-iu} du \\ &= -\frac{1}{2\pi i} \int_{\mathbb{R}} e^{isu} u e^{-u^2} (e^{-itu} - 1) du \\ &= -\frac{1}{2\pi i} \int_{\mathbb{R}} u e^{i(s-t)u} e^{-u^2} du \\ &\quad + \frac{1}{2\pi i} \int_{\mathbb{R}} u e^{isu} e^{-u^2} du \\ &= \Phi(s) - \Phi(s-t), \end{aligned}$$

where

$$\Phi(s) = \frac{1}{2\pi i} \int_{\mathbb{R}} u e^{isu} e^{-u^2} du.$$

We have:

$$\Phi(s) = \frac{1}{2\pi i} \int_{\mathbb{R}} u e^{isu} e^{-u^2} du = \frac{1}{2\pi i} e^{-\frac{s^2}{4}} \int_{\mathbb{R}} u e^{-(u-\frac{is}{2})^2} du = \frac{s}{4\sqrt{\pi}} e^{-\frac{s^2}{4}}.$$

Thus,

$$(T_m 1_{[0,t]})(s) = \frac{1}{4\sqrt{\pi}} \left\{ (t-s) e^{-\frac{(t-s)^2}{4}} + s e^{-\frac{s^2}{4}} \right\}.$$

The support of the function $T_m(I_t)$ is not bounded, and in particular (3.6) is not in force.

When m is bounded, we note that T_m is a translation invariant operator.

We now recall the definitions of the Hermite polynomials and of the Hermite functions. Then, in Proposition 3.6 below we study the action of the operator T_m on Hermite functions.

Definition 3.4. *The Hermite polynomials $\{h_n(x) \mid n \in \mathbb{N}_0\}$ are defined by*

$$h_n(x) \stackrel{\text{def.}}{=} (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} (e^{-\frac{x^2}{2}}), \quad n = 0, 1, 2, \dots$$

Definition 3.5. *The Hermite functions are defined by*

$$\tilde{h}_n(x) \stackrel{\text{def.}}{=} \frac{h_{n-1}(\sqrt{2}x) e^{-\frac{x^2}{2}}}{\pi^{\frac{1}{4}} \sqrt{(n-1)!}}, \quad n = 1, 2, \dots$$

The following proposition the main properties of the Hermite functions which we will need; see [8, p. 349] and the references therein.

Proposition 3.6. [8, p. 349] *The Hermite functions $\{\tilde{h}_n, \ n \in \mathbb{N}\}$ form an orthonormal basis of $\mathbf{L}_2(\mathbb{R})$. Furthermore,*

$$(3.7) \quad |\tilde{h}_n(u)| \leq \begin{cases} C n^{-\frac{1}{12}} & \text{if } |u| \leq 2\sqrt{n}, \\ C e^{-\gamma u^2} & \text{if } |u| > 2\sqrt{n}, \end{cases}$$

where C and $\gamma > 0$ are constants independent of n . Finally, the Fourier transform of the Hermite function is given by

$$(3.8) \quad \widehat{\tilde{h}}_n(u) = \sqrt{2\pi} (-1)^{n-1} \tilde{h}_n(u).$$

Using the previous proposition, we now study the functions $T_m \tilde{h}_n$.

Proposition 3.7. *Assume that the function m satisfies a bound of the type:*

$$(3.9) \quad m(u) \leq \begin{cases} K |u|^{-b} & \text{if } |u| \leq 1, \\ K' & \text{if } |u| > 1, \end{cases}$$

where $b < 2$ and $0 < K, K' < \infty$. Then,

$$(3.10) \quad \left| (T_m \tilde{h}_n)(u) \right| \leq \tilde{C}_1 n^{\frac{5}{12}} + \tilde{C}_2,$$

where \tilde{C}_1 and \tilde{C}_2 are constants independent of n .

Proof: Using (3.8) we have

$$\begin{aligned} |(T_m \tilde{h}_n)(u)| &= \frac{1}{2\pi} \left| \int_{\mathbb{R}} e^{iuy} \widehat{\tilde{h}_n}(y) \sqrt{m(y)} dy \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\tilde{h}_n(y)| \sqrt{m(y)} dy. \end{aligned}$$

We now compute an upper bound for the integral

$$\int_{\mathbb{R}} |\tilde{h}_n(y)| \sqrt{m(y)} dy = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \int_{-\infty}^{-2\sqrt{n}} |\tilde{h}_n(y)| \sqrt{m(y)} dy, \\ I_2 &= \int_{-2\sqrt{n}}^{2\sqrt{n}} |\tilde{h}_n(y)| \sqrt{m(y)} dy, \\ I_3 &= \int_{2\sqrt{n}}^{\infty} |\tilde{h}_n(y)| \sqrt{m(y)} dy. \end{aligned}$$

By (3.7) we have

$$\begin{aligned} I_1 &\leq C \int_{-\infty}^{-2\sqrt{n}} e^{-\gamma y^2} \sqrt{m(y)} dy, \\ I_2 &\leq C n^{-\frac{1}{12}} \int_{-2\sqrt{n}}^{2\sqrt{n}} \sqrt{m(y)} dy, \\ I_3 &\leq C \int_{2\sqrt{n}}^{\infty} e^{-\gamma y^2} \sqrt{m(y)} dy. \end{aligned}$$

We have:

$$\begin{aligned} \int_{-2\sqrt{n}}^{2\sqrt{n}} \sqrt{m(y)} dy &= \int_{-2\sqrt{n}}^{-1} \sqrt{m(y)} dy + \int_{-1}^1 \sqrt{m(y)} dy + \\ &\quad + \int_1^{2\sqrt{n}} \sqrt{m(y)} dy \\ &\leq \sqrt{K'} \int_{-2\sqrt{n}}^{-1} dy + \sqrt{K} \int_{-1}^1 |y|^{-\frac{b}{2}} dy + \\ &\quad + \sqrt{K'} \int_1^{2\sqrt{n}} dy \\ &= 2\sqrt{K'}(2\sqrt{n} - 1) + 4\sqrt{K} \frac{1}{2-b}, \end{aligned}$$

so that we get

$$I_2 \leq C_1 n^{\frac{5}{12}} + C_2 n^{-\frac{1}{12}}.$$

Furthermore,

$$\begin{aligned} \int_{2\sqrt{n}}^{\infty} e^{-\gamma y^2} \sqrt{m(y)} dy &\leq \sqrt{K'} \int_{2\sqrt{n}}^{\infty} e^{-\gamma y^2} dy \\ &\leq \sqrt{K'} \int_{\mathbb{R}} e^{-\gamma y^2} dy \\ &= 2\sqrt{K'} \sqrt{\frac{\pi}{\gamma}} = C_3. \end{aligned}$$

Finally we get

$$\begin{aligned} \left| (T_m \tilde{h}_n)(y) \right| &\leq \frac{1}{\sqrt{2\pi}} (C_1 n^{\frac{5}{12}} + C_2 n^{-\frac{1}{12}} + 2C_3) \\ &\leq \tilde{C}_1 n^{\frac{5}{12}} + \tilde{C}_2 \end{aligned}$$

for appropriate \tilde{C}_1 and \tilde{C}_2 . □

Lemma 3.8. *The function $T_m \tilde{h}_k$ is uniformly continuous for every $k \in \mathbb{N}$. More precisely, it holds that*

$$(3.11) \quad \left| (T_m \tilde{h}_k)(t) - (T_m \tilde{h}_k)(s) \right| \leq |t - s| \left\{ \tilde{C}_1 k^{\frac{11}{12}} + \tilde{C}_2 \right\},$$

where \tilde{C}_1 and \tilde{C}_2 are constant independent of k .

Proof: Let $t, s \in \mathbb{R}$. We have

$$(T_m \tilde{h}_k)(t) - (T_m \tilde{h}_k)(s) = \frac{1}{\sqrt{2\pi}} (-1)^{k-1} \int_{\mathbb{R}} (e^{-iut} - e^{-ius}) \sqrt{m(u)} \tilde{h}_k(u) du.$$

Taking into account that

$$|e^{-iut} - e^{-ius}| \leq |u(t - s)|,$$

we get

$$\left| (T_m \tilde{h}_k)(t) - (T_m \tilde{h}_k)(s) \right| \leq \frac{|t - s|}{\sqrt{2\pi}} \int_{\mathbb{R}} |u| \sqrt{m(u)} \left| \tilde{h}_k(u) \right| du.$$

To conclude the proof, it suffices to show that $\int_{\mathbb{R}} |u| \sqrt{m(u)} \left| \tilde{h}_k(u) \right| du < \infty$. By (3.9) and (3.7) we have

$$\begin{aligned} \int_{\mathbb{R}} |u| \sqrt{m(u)} \left| \tilde{h}_k(u) \right| du &\leq Ak^{-\frac{1}{12}} \int_{|u| \leq 1} |u|^{1-\frac{b}{2}} du + \\ &\quad + Bk^{-\frac{1}{12}} \int_{1 < |u| \leq 2\sqrt{k}} |u| du + \\ &\quad + C \int_{|u| > 2\sqrt{k}} |u| e^{-\gamma u^2} du \\ &\leq \tilde{C}_1 k^{\frac{11}{12}} + \tilde{C}_2, \end{aligned}$$

where all the constants are independent of k . \square

We conclude this section with a remark.

Remark 3.9. *When*

$$\int_{\mathbb{R}} \frac{\ln m(u)}{u^2 + 1} du > -\infty,$$

*the function m admits a factorization $m(u) = |h(u)|^2$, where h is an outer function. One can define an operator \tilde{T}_m through $\tilde{T}_m f \stackrel{\text{def}}{=} \hat{h} * f$ rather than the operator T_m . We will not pursue this direction here.*

4. THE WHITE NOISE SPACE AND THE BROWNIAN MOTION

In this section we review the construction of the white noise space and recall some results related to the Brownian motion. We refer the reader to [21], [22], [30] and [23] for additional information and references. To build the white noise space one considers the subspace $\mathcal{S}_{\mathbb{R}}(\mathbb{R})$ of the Schwartz space which consists of *real valued* functions. Denote by $\mathcal{S}_{\mathbb{R}}(\mathbb{R})'$ its dual. Let \mathcal{F} be the σ -algebra of Borel sets in the space $\mathcal{S}_{\mathbb{R}}(\mathbb{R})'$. The function

$$K(s_1 - s_2) = \exp(-\|s_1 - s_2\|_{\mathbf{L}_2(\mathbb{R})}^2 / 2)$$

is positive in $\mathcal{S}_{\mathbb{R}}(\mathbb{R})$ in the sense of reproducing kernels since

$$\begin{aligned} \exp(-\|s_1 - s_2\|_{\mathbf{L}_2(\mathbb{R})}^2 / 2) &= \exp(-\|s_1\|_{\mathbf{L}_2(\mathbb{R})}^2 / 2) \times \\ &\quad \times \exp\langle s_1, s_2 \rangle_{\mathbf{L}_2(\mathbb{R})} \times \exp(-\|s_2\|_{\mathbf{L}_2(\mathbb{R})}^2 / 2). \end{aligned}$$

The space $\mathcal{S}_{\mathbb{R}}(\mathbb{R})$ is nuclear, and therefore the Bochner–Minlos theorem (see for instance [22, Appendix A, p. 193]) implies that there exists a

probability measure P on $(\mathcal{S}_{\mathbb{R}}(\mathbb{R})', \mathcal{F})$ such that, for all $s \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$,

$$(4.1) \quad E(e^{iQ_s(s')}) \stackrel{\text{def.}}{=} \int_{\mathcal{S}'(\mathbb{R})} e^{iQ_s(s')} dP(s') = e^{-\frac{\|s\|_{\mathbf{L}_2(\mathbb{R})}^2}{2}},$$

where Q_s denotes the linear functional $Q_s(s') = \langle s', s \rangle_{\mathcal{S}_{\mathbb{R}}(\mathbb{R})', \mathcal{S}_{\mathbb{R}}(\mathbb{R})}$; see [8, (2.1) p. 348], [19, (2.3) p. 303]. Note that Q_s is the canonical isomorphism of the Schwartz space $\mathcal{S}_{\mathbb{R}}(\mathbb{R})$ onto its bidual; see [24, p. 7]. Definition (4.1) implies in particular that

$$(4.2) \quad E(Q_s) = 0 \quad \text{and} \quad E(Q_s^2) = \|s\|_{\mathbf{L}_2(\mathbb{R})}^2.$$

In view of (4.2), the map

$$(4.3) \quad s \rightarrow Q_s$$

is an isometry from the real Hilbert space $\mathcal{S}_{\mathbb{R}}(\mathbb{R}) \subset \mathbf{L}_2(\mathbb{R})$ into the real Hilbert space $\mathbf{L}_2(\mathcal{S}_{\mathbb{R}}(\mathbb{R})', \mathcal{F}, dP)$. It extends to an isometry from $\mathbf{L}_2(\mathbb{R})$ into $\mathbf{L}_2(\mathcal{S}_{\mathbb{R}}(\mathbb{R})', \mathcal{F}, dP)$, and we define for $f \in \mathbf{L}_2(\mathbb{R})$

$$(4.4) \quad Q_f(s') \stackrel{\text{def.}}{=} \lim_{n \rightarrow \infty} Q_{f_n}(s'),$$

where the limit is in $\mathbf{L}_2(\mathcal{S}_{\mathbb{R}}(\mathbb{R})', \mathcal{F}, dP)$ and where $f_n \rightarrow f$ in $\mathbf{L}_2(\mathbb{R})$. The limit is easily shown not to depend on (f_n) .

In the sequel we consider complex-valued functions. The map (4.3) extends to an isometry between the complexified spaces of $\mathbf{L}_2(\mathbb{R})$ and $\mathbf{L}_2(\mathcal{S}'(\mathbb{R}), \mathcal{F}, dP)$. See for instance [9, pp. V4-V5] for the complexification of Hilbert spaces.

The triplet $\mathbf{L}_2(\mathcal{S}_{\mathbb{R}}(\mathbb{R})', \mathcal{F}, P)$ is called the white noise space. In accordance with the notation standard in probability theory, we set

$$\Omega = \mathcal{S}_{\mathbb{R}}(\mathbb{R})',$$

and denote by

$$\mathcal{W} = \mathbf{L}_2(\Omega, \mathcal{F}, P).$$

the complexified space of $\mathbf{L}_2(\mathcal{S}_{\mathbb{R}}(\mathbb{R})', \mathcal{F}, P)$.

The Brownian motion is a family $\{B(t, \omega)\}$ of random variables in the white noise space with the following property:

- (1) $B(0, \omega) = 0$ almost surely with respect to P .
- (2) $\{B(t, \omega)\}$ is a Gaussian stochastic process with mean zero and $B(t, \omega)$ and $B(s, \omega)$ have the covariance $\min(t, s)$.
- (3) $s \rightarrow B(s, \omega)$ is a continuous for almost all ω with respect to P .

Define the stochastic process

$$\tilde{B}(t, \omega) = Q_{I_t}(\omega), \quad t \in \mathbb{R}.$$

Then, for $t, s \geq 0$,

$$E(\tilde{B}(t, \omega)\tilde{B}(s, \omega)^*) = \langle I_t, I_s \rangle_{\mathbf{L}_2(\mathbb{R})} = \int_{\mathbb{R}} I_t(u)(I_s(u))^* du = \min(t, s).$$

By Kolmogorov's continuity Theorem the process $\{\tilde{B}(t, \omega)\}$ has a continuous version $\{B(t, \omega)\}$, with is a Brownian motion. For $F \in \mathcal{W}$ we now recall Wiener-Ito chaos expansion. In the stochastic process literature this expansion is for real valued functions. We write it for the complexification of the underlying Hilbert spaces. This creates no technical problem.

The white noise probability space \mathcal{W} admits a special orthonormal basis $\{H_\alpha\}$, indexed by the set ℓ (defined by (1.8)). We will not recall the definition of this basis here, and refer to [22, Definition 2.2.1 p. 19].

Proposition 4.1. (*Wiener-Ito chaos expansion*) *Every $F \in \mathcal{W}$ can be written as*

$$F = \sum_{\alpha \in \ell} c_\alpha H_\alpha,$$

with $\alpha \in \ell$, $c_\alpha \in \mathbb{C}$, and

$$\|F\|_{\mathcal{W}}^2 = \sum_{\alpha \in \ell} \alpha! |c_\alpha|^2 < \infty,$$

where $\alpha! = \alpha_1! \alpha_2! \alpha_3! \cdots$ and

$$(4.5) \quad H_\alpha(\omega) \stackrel{\text{def.}}{=} \prod_{k=1}^{\infty} h_{\alpha_k} \left(Q_{\tilde{h}_k}(\omega) \right), \quad \omega \in \Omega.$$

5. THE KONDRATIEV SPACE AND THE WICK PRODUCT

The Wick product is defined through:

Definition 5.1. *Let $\alpha, \beta \in \ell$, then*

$$H_\alpha \diamond H_\beta = H_{\alpha+\beta}.$$

Definition 5.2. *Let F, G be two elements in \mathcal{W}*

$$F = \sum_{\alpha} a_{\alpha \in \ell} H_\alpha, \quad \text{and} \quad G = \sum_{\alpha \in \ell} b_\alpha H_\alpha,$$

where $\alpha \in \ell$, $a_\alpha, b_\alpha \in \mathbb{C}$ and $a_\alpha, b_\alpha \neq 0$ for only a finite number of indexes α . The Wick product of F and G is defined by

$$(F \diamond G)(\omega) = \sum_{\alpha, \beta \in \ell} a_\alpha b_\beta H_{\alpha+\beta}(\omega) = \sum_{\gamma} \left(\sum_{\gamma=\alpha+\beta} a_\alpha b_\beta \right) H_\gamma(\omega).$$

This product can be shown to be independent of the basis $\{H_\alpha\}_{\alpha \in \ell}$, see [22, Appendix D, p. 209], but it is not defined for all pairs of elements in the white noise space. See [22].

The Kondratiev space S_{-1} seems to be the most convenient space within which the Wick product is well defined. It is a space of distributions. We first recall on which space of test functions its elements operate.

Definition 5.3. *The Kondratiev space S_1 of stochastic test functions consists of the elements in the form $f = \sum_{\alpha \in \ell} a_\alpha H_\alpha \in \mathcal{W}$ such that*

$$\sum_{\alpha \in \ell} |a_\alpha|^2 (\alpha!)^2 (2\mathbb{N})^{k\alpha} < \infty, \quad k = 1, 2, \dots,$$

and where

$$(2\mathbb{N})^\alpha \stackrel{\text{def.}}{=} 2^{\alpha_1} (2 \cdot 2)^{\alpha_2} (2 \cdot 3)^{\alpha_3} \dots, \quad \alpha \in \ell.$$

Definition 5.4. *The Kondratiev space S_{-1} of stochastic distributions consists of the elements in the form $F = \sum_{\alpha \in \ell} b_\alpha H_\alpha$ with the property that*

$$\sum_{\alpha \in \ell} |b_\alpha|^2 (2\mathbb{N})^{-q\alpha} < \infty,$$

for some $q \in \mathbb{N}$.

S_{-1} can be identified with the dual of S_1 and the action of $F \in S_{-1}$ on $f = \sum_{\alpha \in \ell} a_\alpha H_\alpha \in S_1$ is given by

$$(5.1) \quad \langle F, f \rangle_{S_{-1}, S_1} \stackrel{\text{def.}}{=} \sum_{\alpha \in \ell} \alpha! a_\alpha b_\alpha.$$

We also note the following: let $\alpha \in \ell$. By (4.5), using a Wick product calculation, we have

$$H_\alpha(\omega) = \prod_{k=1}^{\infty} \left(Q_{\tilde{h}_k}(\omega) \right)^{\diamond \alpha_k}$$

for $\alpha = \epsilon^{(k)} = (0, 0, \dots, 0, 1, 0, \dots)$, $\alpha_i = 0$ for $i \neq k$ and $\alpha_k = 1$ we get

$$H_{\epsilon^{(k)}} = Q_{\tilde{h}_k} = \int_{\mathbb{R}} \tilde{h}_k(t) dB(t).$$

We now review the main results associated with the Wick product and the Hermite transform.

A key property of the basis $\{H_\alpha, \alpha \in \ell\}$ is the following: define a map \mathbf{I} such that

$$\mathbf{I}(H_\alpha) = z^\alpha,$$

where $\alpha \in \ell$, $z = (z_1, z_2, \dots) \in \mathbb{C}^\mathbb{N}$ (the set of all sequences of complex numbers) and

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \dots.$$

Then

$$\mathbf{I}(H_\alpha \diamond H_\beta) = \mathbf{I}(H_\alpha) \mathbf{I}(H_\beta).$$

The map \mathbf{I} is called the Hermite transform.

We note that the spaces S_{-1} and S_1 are closed under the Wick product; see [22, lemma 2.4.4, p 42].

Definition 5.5. Let $F = \sum_{\alpha \in \ell} a_\alpha H_\alpha \in S_{-1}$. Then the Hermite transform of F , denoted by $\mathbf{I}(F)$ or \tilde{F} , is defined by

$$\mathbf{I}(F)(z) = \tilde{F}(z) = \sum_{\alpha \in \ell} a_\alpha z^\alpha.$$

Proposition 5.6. [22, Proposition 2.6.6, p. 59] Let $F, G \in S_{-1}$. Then

$$\mathbf{I}(F \diamond G)(z) = (\mathbf{I}(F)(z)) \cdot (\mathbf{I}(G)(z)).$$

6. THE m -BROWNIAN MOTION ASSOCIATED WITH T_m

In this section we define the m -Brownian motion associated with the operator T_m defined in (3.2), and the analogue of the Wiener-Ito chaos expansion (see Proposition 4.1). Using Lemma 3.7 we have $T_m I_t \in \mathbf{L}_2(\mathbb{R})$ and by the expansion in $\mathbf{L}_2(\mathbb{R})$ in term of the Hermite functions \tilde{h}_n we obtain

$$T_m I_t = \sum_{k=1}^{\infty} \langle T_m I_t, \tilde{h}_k \rangle \tilde{h}_k,$$

where

$$\langle T_m I_t, \tilde{h}_k \rangle = \langle I_t, T_m \tilde{h}_k \rangle = \int_{\mathbb{R}} I_t(y) (T_m \tilde{h}_k)(y) dy.$$

Definition 6.1. The process $\{\tilde{B}_m(t, \omega), t \in \mathbb{R}\}$ defined through

$$\tilde{B}_m(t, \omega) \stackrel{\text{def.}}{=} Q_{T_m I_t}(\omega),$$

where $t \in \mathbb{R}$ and $\omega \in \Omega$ will be called the m -Brownian motion associated with m .

As already noted in Section 3, when the function m is given by (3.5) (or, equivalently, the function r is given by (1.5)), the operator T_m is equal to the operator M_H defined in [19] and [8]. Then, B_m reduces to the fractional Brownian motion with Hurst parameter $H \in (0, 1)$.

Lemma 6.2. *The m -Brownian motion has the following properties:*

- (1) $E(\tilde{B}_m(t, \omega)\tilde{B}_m(s, \omega)^*) = K_r(s, t).$
- (2) $E\left(\left|\tilde{B}_m(t, \omega) - \tilde{B}_m(s, \omega)\right|^2\right) = 2 \operatorname{Re} r(t - s).$
- (3) $\operatorname{Re} r(t) \leq C_1 t^2 + C_2 t$

for some positive constants C_1 and C_2 .

Proof: To prove item (1) we note that

$$\begin{aligned}
E(\tilde{B}_m(t, \omega)\tilde{B}_m(s, \omega)^*) &= \langle T_m I_t, T_m I_s \rangle_{\mathbf{L}_2(\mathbb{R})} \\
&= \int_{\mathbb{R}} (T_m I_t)(u)((T_m I_t)(u))^* du \\
&= \int_{\mathbb{R}} (\widehat{T_m I_t})(u)(\widehat{T_m I_s}(u))^* du \\
&= \int_{\mathbb{R}} m(u)(\widehat{1_{[0,t]}})(u) \left(\widehat{1_{[0,s]}}(u) \right)^* du \\
&= \int_{\mathbb{R}} m(u) \left\{ \int_0^t e^{-iux} dx \right\} \left\{ \int_0^s e^{iuy} dy \right\} du \\
&= \int_{\mathbb{R}} \frac{e^{-itu} - 1}{u} \frac{e^{isu} - 1}{u} m(u) du \\
&= K_r(s, t).
\end{aligned}$$

The proof of the second statement is carried out by direct computations:

$$\begin{aligned}
&E\left(\left|\tilde{B}_m(t, \omega) - \tilde{B}_m(s, \omega)\right|^2\right) \\
&= E((\tilde{B}_m(t, \omega) - \tilde{B}_m(s, \omega))(\tilde{B}_m(t, \omega) - \tilde{B}_m(s, \omega))^*) \\
&= E\left\{\tilde{B}_m(t, \omega)\tilde{B}_m(t, \omega)^* - \tilde{B}_m(t, \omega)\tilde{B}_m(s, \omega)^* \right. \\
&\quad \left. - \tilde{B}_m(s, \omega)\tilde{B}_m(t, \omega)^* + \tilde{B}_m(s, \omega)\tilde{B}_m(s, \omega)^*\right\}.
\end{aligned}$$

Using the first statement we get

$$\begin{aligned}
&= E(\tilde{B}_m(t, \omega) \tilde{B}_m(t, \omega)^*) - E(\tilde{B}_m(t, \omega) \tilde{B}_m(s, \omega)^*) \\
&\quad - E(\tilde{B}_m(s, \omega) \tilde{B}_m(t, \omega)^*) + E(\tilde{B}_m(s, \omega) \tilde{B}_m(s, \omega)^*) \\
&= \{K_r(t, t) - K_r(s, t) - K_r(t, s) + K_r(s, s)\} \\
&= (r(t) + r(t)^* - (r(s) + r(t)^* - r(s - t)) \\
&\quad - (r(t) + r(s)^* - r(t - s)) + r(s) + r(s)^*) \\
&= (r(t - s) + r(t - s)^*) \\
&= 2 \operatorname{Re} r(t - s).
\end{aligned}$$

Finally, recall that we have

$$\operatorname{Re} r(t) = \int_{\mathbb{R}} \left\{ 1 - \cos(tu) \right\} \frac{m(u)}{u^2} du.$$

Then, when $m(u)$ satisfies (3.10) we get

$$\begin{aligned}
\operatorname{Re} r(t) &\leq 2 \left\{ K \int_0^1 \frac{|1 - \cos(tu)|}{u^{2+b}} du + K' \int_1^\infty \frac{|1 - \cos(tu)|}{u^2} du \right\} \\
&= 2 \left\{ K \int_0^1 \frac{2 \sin^2(\frac{tu}{2})}{u^{2+b}} du + K' \int_1^\infty \frac{|1 - \cos(tu)|}{u^2} du \right\}.
\end{aligned}$$

But

$$\int_0^1 \frac{2 \sin^2(\frac{tu}{2})}{u^{2+b}} du \leq t^2 \int_0^1 \frac{u^2}{2u^{2+b}} du = \frac{t^2}{2} \int_0^1 \frac{1}{u^b} du = \frac{t^2}{2(1-b)}$$

since for $t \in [0, 1]$ we get $tu \in [0, 1]$ and $\sin^2(\frac{tu}{2}) \leq \frac{(tu)^2}{4}$. Furthermore,

$$\begin{aligned}
\int_1^\infty \frac{|1 - \cos(tu)|}{u^2} du &\leq \int_0^\infty \frac{|1 - \cos(tu)|}{u^2} du \\
&= t \int_0^\infty \frac{|1 - \cos v|}{v^2} dv \\
&= t \int_0^1 \frac{|1 - \cos v|}{v^2} dv + t \int_1^\infty \frac{|1 - \cos v|}{v^2} dv \\
&\leq t \int_0^1 \frac{2 \sin^2(\frac{v}{2})}{v^2} dv + 2t \int_1^\infty \frac{1}{v^2} dv \\
&\leq \frac{t}{2} \int_0^1 \frac{v^2}{v^2} dv + 2t \int_1^\infty \frac{1}{v^2} dv \\
&= \frac{t}{2} + 2t = \frac{5t}{2}.
\end{aligned}$$

Thus,

$$\operatorname{Re} r(t) \leq |\operatorname{Re} r(t)| \leq 2 \left\{ K \frac{t^2}{2(1-b)} + K' \frac{5t}{2} \right\} = C_1 t^2 + C_2 t.$$

□

The next step is to show that $\{\tilde{B}_m(t, \omega), t \in \mathbb{R}\}$ meets the criterion of Kolmogorov Theorem concerning the existence of a continuous version of a given stochastic process. Using the fact (see for instance [24, p.5] with $p = 2n$) that

$$(6.1) \quad E\left(\left|\tilde{B}_m(t, \omega)\right|^{2n}\right) = \kappa(2n)^{2n} \left(E\left(\left|\tilde{B}_m(t, \omega)\right|^2\right)\right)^{\frac{2n}{2}}$$

where

$$\kappa(2n) = \sqrt{2} \left(\frac{\Gamma(\frac{2n+1}{2})}{\sqrt{\pi}} \right)^{\frac{1}{2n}} = \sqrt{2} \left(\frac{2n!}{4^n n!} \right)^{\frac{1}{2n}}$$

we have

$$E\left(\left|\tilde{B}_m(t, \omega) - \tilde{B}_m(s, \omega)\right|^4\right) = \kappa(4)^4 E\left(\left|\tilde{B}_m(t, \omega) - \tilde{B}_m(s, \omega)\right|^2\right)^2.$$

By (2), (3) we get

$$\begin{aligned} \kappa(4)^4 E\left(\left|\tilde{B}_m(t, \omega) - \tilde{B}_m(s, \omega)\right|^2\right)^2 &= \kappa(4)^4 (\operatorname{Re} r(t-s))^2 \\ &\leq \kappa(4)^4 (C_1(t-s)^2 + C_2(t-s))^2 = (t-s)^2 (A + B(t-s))^2, \end{aligned}$$

for $t-s \in [0, 1]$. By Kolmogorov's continuity theorem the process $\tilde{B}_m(t, \omega)$ has a continuous version $B_m(t, \omega)$ where $t \in [0, 1]$, which we will define as the *m-Brownian motion associated with T_m* . One can show in a similar way that a continuous version exists on every finite interval.

Proposition 6.3. *$B_m(t)$ is a Gaussian random variable with*

$$E(B_m^n(t, \omega)) = \begin{cases} 0, & \text{if } n = 2k-1 \\ \frac{(2k)!}{2^k k!} \|T_m I_t\|^{2k}, & \text{if } n = 2k \end{cases}$$

for $k = 1, 2, \dots$

Proof: By (4.1) with $B_m(t, \omega) = Q_{T_m I_t}(\omega)$, we have

$$E(\exp(iB_m(t, \omega))) = e^{-\frac{\|T_m I_t\|^2}{2}},$$

and therefore $B_m(t, \omega)$ is a Gaussian random variable. We now verify that

$$E\left(\sum_{n=0}^{\infty} \frac{i^n}{n!} B_m^n(t, \omega)\right) = \sum_{n=0}^{\infty} \frac{i^n}{n!} E(B_m^n(t, \omega)).$$

Since $e^{-\frac{\|T_m I_t\|^2}{2}} \in \mathbb{R}$, we have that:

$$E\left(\sum_{n=0}^{\infty} \frac{i^n}{n!} B_m^n(t, \omega)\right) = E\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} B_m^{2n}(t, \omega)\right).$$

Since $B_m(t, \omega)$ is a centered Gaussian random variable, we have that $E(B_m^{2k-1}(t, \omega)) = 0$ for $k = 1, 2, \dots$, and we get

$$\sum_{n=0}^{\infty} \frac{i^n}{n!} E(B_m^n(t, \omega)) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} E(B_m^{2n}(t, \omega)).$$

We have to verify

$$E\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} B_m^{2n}(t, \omega)\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} E(B_m^{2n}(t, \omega)).$$

Let $\epsilon \in \mathbb{R}$. Using (6.1) we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{|\epsilon|^{2n} E(|B_m(t, \omega)|^{2n})}{2n!} &= \sum_{n=0}^{\infty} \frac{|\epsilon|^{2n} \kappa(2n)^{2n} (E|B_m(t, \omega)|^2)^n}{2n!} \\ &= \sum_{n=0}^{\infty} \frac{|\epsilon|^{2n} (E|B_m(t, \omega)|^2)^n}{2^n n!} < \infty. \end{aligned}$$

We can thus use the dominated convergence theorem, to obtain that

$$\begin{aligned} E\left(\sum_{k=0}^{\infty} \epsilon^k \frac{i^k}{k!} B_m^k(t, \omega)\right) &= \sum_{k=0}^{\infty} \epsilon^k \frac{i^k}{k!} E(B_m^k(t, \omega)) \\ &= \sum_{\ell=0}^{\infty} (-1)^\ell \epsilon^{2\ell} \frac{\|T_m I_t\|^{2\ell}}{2^\ell \ell!}. \end{aligned}$$

The proof is completed by comparing the powers of ϵ on both sides. \square

Remark 6.4. In view of (1.4) we have

$$\|T_m I_t\|^2 = K_r(t, t).$$

Remark 6.5. Since for any $t \in \mathbb{R}$, $B_m(t)$ is written as a weighted sum of the $\{H_\alpha, \alpha \in \ell\}$ (for an explicit expression, see (7.3) below), in turn being jointly Gaussian random variables, it follows that $\{B_m(t), t \in \mathbb{R}\}$ is a Gaussian process.

The following proposition will be used in the sequel of this paper, where, as already noted, we develop the stochastic analysis associated with the processes B_m .

Proposition 6.6. *Let $f \in \text{dom}(T_m)$ and $n \in \mathbb{N}$. It holds that:*

$$Q_{T_m f}^{\diamond n}(\omega) = n! \sum_{k=\lceil \frac{n}{2} \rceil}^n \left(-\frac{1}{2}\right)^{n-k} \frac{Q_{T_m f}^{2k-n}(\omega)}{(2k-n)!} \frac{(\|T_m f\|^2)^{n-k}}{(n-k)!}.$$

In particular, for $f = I_t$, it holds that

$$B_m^{\diamond n}(t) = n! \sum_{k=\lceil \frac{n}{2} \rceil}^n \left(-\frac{1}{2}\right)^{n-k} \frac{B_m^{2k-n}(t)}{(2k-n)!} \frac{(\|T_m I_t\|^2)^{n-k}}{(n-k)!}.$$

Proof: Let $\epsilon \in \mathbb{R}$ then

$$\exp^{\diamond}(Q_{\epsilon T_m f}(\omega)) = \sum_{n=0}^{\infty} \frac{(Q_{\epsilon T_m f}(\omega))^{\diamond n}}{n!} = \sum_{n=0}^{\infty} \frac{\epsilon^n (Q_{T_m f}(\omega))^{\diamond n}}{n!}.$$

By [24, Theorem 3.33, p. 32], we have

$$\begin{aligned} \exp^{\diamond}(Q_{\epsilon T_m f}(\omega)) &= \exp(Q_{\epsilon T_m f}(\omega) - \frac{1}{2} \|\epsilon T_m f\|^2) \\ &= \sum_{k=0}^{\infty} \frac{(\epsilon Q_{T_m f}(\omega) - \frac{1}{2} \epsilon^2 \|T_m f\|^2)^k}{k!} \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^k \epsilon^{2k-j} \frac{(Q_{T_m f}(\omega))^j}{j!} \frac{(-\frac{1}{2} \|T_m f\|^2)^{k-j}}{(k-j)!}. \end{aligned}$$

Hence,

$$\sum_{n=0}^{\infty} \frac{\epsilon^n (Q_{T_m f}(\omega))^{\diamond n}}{n!} = \sum_{k=0}^{\infty} \sum_{j=0}^k \epsilon^{2k-j} \frac{(Q_{T_m f}(\omega))^j}{j!} \frac{(-\frac{1}{2} \|T_m f\|^2)^{k-j}}{(k-j)!},$$

and comparing the powers of ϵ leads to:

$$\frac{(Q_{T_m f}(\omega))^{\diamond n}}{n!} = \sum_{k=\lceil \frac{n}{2} \rceil}^n \left(-\frac{1}{2}\right)^{n-k} \frac{(Q_{T_m f}(\omega))^{2k-n}}{(2k-n)!} \frac{(\|T_m f\|^2)^{n-k}}{(n-k)!}.$$

□

7. THE m -WHITE NOISE

One important aspect of the white noise space theory is that the Brownian motion admits a derivative, which belongs to the Hida space $(S)^*$ (the definition of which we do not recall in this paper), and in particular in the Kondratiev space S_{-1} . See [22, p. 53]. In this section we prove that this result still holds for the m -Brownian motion. For the next definition, see also [22, Definition 2.5.5, p. 49], where the integral is defined to be an element in the Hida space $(S)^*$.

Definition 7.1. Suppose that $Z : \mathbb{R} \rightarrow S_{-1}$ is a given function with the property that

$$\langle Z(t), f \rangle \in \mathbf{L}_1(\mathbb{R}, dt)$$

for all $f \in S_1$. Then $\int_{\mathbb{R}} Z(t)dt$ is defined to be the unique element of S_{-1} such that

$$\left\langle \int_{\mathbb{R}} Z(t)dt, f \right\rangle = \int_{\mathbb{R}} \langle Z(t), f \rangle dt$$

for all $f \in S_1$.

In view of Lemma 3.8 the coefficients of the expansion (7.1) below are continuous functions, and not merely elements of $\mathbf{L}_2(\mathbb{R})$.

Definition 7.2. The m -white noise $W_m(t)$ is defined by

$$(7.1) \quad W_m(t) = \sum_{k=1}^{\infty} \left(T_m \tilde{h}_k \right) (t) H_{\epsilon^{(k)}}.$$

Theorem 7.3. For every real t we have that $W_m(t) \in S_{-1}$, and it holds that

$$(7.2) \quad B_m(t) = \int_0^t W_m(s)ds, \quad t \in \mathbb{R}.$$

Proof: Let $q \geq 2 \in \mathbb{N}$. Then, using (3.10), we have:

$$\sum_{k=1}^{\infty} \left| \left(T_m \tilde{h}_k \right) (t) \right|^2 (2k)^{-q} \leq \sum_{k=1}^{\infty} (\tilde{C}_1 k^{\frac{5}{12}} + \tilde{C}_2)^2 (2k)^{-q} < \infty,$$

and so $W_m(t) \in S_{-1}$. We now prove (7.2). By construction, $B_m(t) \in \mathcal{W}$ for every $t \in \mathbb{R}$, and we can write

$$(7.3) \quad B_m(t) = \sum_{k=1}^{\infty} b_k(t) H_{\epsilon^{(k)}},$$

where

$$b_k(t) = \int_0^t \left(T_m \tilde{h}_k \right) (s) ds,$$

with the convergence in the topology of \mathcal{W} . We want to show that, for every $f \in S_1$, we have

$$\langle B_m(t), f \rangle_{S_{-1}, S_1} = \int_0^t \langle W_m(u), f \rangle_{S_{-1}, S_1} du,$$

where $\langle \cdot, \cdot \rangle_{S_{-1}, S_1}$ denotes the duality between S_1 and S_{-1} (see (5.1)). To that purpose, let $q \geq 2 \in \mathbb{N}$. By using the estimate (3.10), then, with $f = \sum_{\alpha \in \ell} f_\alpha H_\alpha$ we have for $u \in [0, t]$ (and in fact for every $u \geq 0$),

$$\begin{aligned} \sum_{k=1}^{\infty} |T_m \tilde{h}_k(u) f_k| &= \sum_{k=1}^{\infty} |T_m \tilde{h}_k(u)| (2k)^{-q} (2k)^q |f_k| \\ &\leq \left(\sum_{k=1}^{\infty} (\tilde{C}_1 n^{\frac{5}{12}} + \tilde{C}_2)^2 (2k)^{-2q} \right)^{\frac{1}{2}} \cdot \left(\sum_{k=1}^{\infty} (2k)^{2q} |f_k|^2 \right)^{\frac{1}{2}} \\ &< \infty, \end{aligned}$$

since $f \in S_1$. Therefore the series

$$\sum_{k=1}^{\infty} |T_m \tilde{h}_k(u) f_k|$$

converges absolutely. Using the dominated convergence theorem we can write

$$\begin{aligned} \int_0^t \langle W_m(u), f \rangle_{S_{-1}, S_1} du &= \int_0^t \left(\sum_{k=1}^{\infty} T_m \tilde{h}_k(u) f_k \right) du \\ &= \sum_{k=1}^{\infty} \left(\int_0^t T_m \tilde{h}_k(u) du \right) f_k \\ &= \langle B_m(t), f \rangle_{S_{-1}, S_1}. \end{aligned}$$

□

We now show that, conversely,

$$B_m(t)' = W_m(t),$$

in the sense of S_{-1} -processes; see [22, p. 77] and further below in the current section. In the following statements, the set $K_q(\delta)$ is defined by

$$K_q(\delta) = \{z \in \mathbb{C}^{\mathbb{N}} : \sum_{\alpha \in \ell} |z^\alpha|^2 (2\mathbb{N})^{q\alpha} < \delta^2\}.$$

See [22, Definition 2.6.4 p. 59].

Proposition 7.4. *The function $\mathbf{I}(W_m(t))(z)$ is bounded for $(t, z) \in \mathbb{R} \times K_2(\delta)$.*

Proof: Write

$$W_m(t) = \sum_{k=1}^{\infty} (T_m \tilde{h}_k)(t) Q_{\tilde{h}_k}.$$

Taking the Hermite transform we have

$$\mathbf{I}(W_m(t))(z) = \sum_{k=1}^{\infty} (T_m \tilde{h}_k)(t) z_k.$$

Thus, for every $q \geq 2 \in \mathbb{N}$ and using (3.10) we have:

$$\begin{aligned} |\mathbf{I}(W_m(t))| &= \left| \sum_{k=1}^{\infty} (T_m \tilde{h}_k)(t) z_k \right| \\ &= \left| \sum_{k=1}^{\infty} (T_m \tilde{h}_k)(t) (2k)^{\frac{q}{2}} (2k)^{-\frac{q}{2}} z_k \right| \\ &= \left(\sum_{k=1}^{\infty} \left| (T_m \tilde{h}_k)(t) \right|^2 (2k)^q \right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} (2k)^{-q} |z^{\epsilon_k}|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{k=1}^{\infty} \left\{ \tilde{C}_1 k^{\frac{5}{12}} + \tilde{C}_2 \right\}^2 (2k)^q \right)^{\frac{1}{2}} \left(\sum_{\alpha \in \ell} (2\mathbb{N})^{-q\alpha} |z^\alpha|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

The first sum converges when $q \geq 2$ and the second converges since $z \in K_2(\delta)$. We conclude that the function $\mathbf{I}(W_m(t))(z)$ is bounded for any pair $(t, z) \in \mathbb{R} \times K_2(\delta)$. \square

Theorem 7.5. *The function $\mathbf{I}(W_m(t))(z)$ is uniformly continuous in t for $z \in K_4(\delta)$.*

Proof: Using the Cauchy-Schwartz inequality, we have:

$$\begin{aligned} |\mathbf{I}(W_m(t))(z) - \mathbf{I}(W_m(s))(z)| &= \left| \sum_{k=1}^{\infty} \left\{ (T_m \tilde{h}_k)(t) - (T_m \tilde{h}_k)(s) \right\} z_k \right| \\ &= \left| \sum_{k=1}^{\infty} \left\{ (T_m \tilde{h}_k)(t) - (T_m \tilde{h}_k)(s) \right\} (2k)^{-\frac{q}{2}} (2k)^{\frac{q}{2}} z_k \right| \\ &\leq \left(\sum_{k=1}^{\infty} \left| \left\{ (T_m \tilde{h}_k)(t) - (T_m \tilde{h}_k)(s) \right\} \right|^2 (2k)^{-q} \right)^{\frac{1}{2}} \times \\ &\quad \times \left(\sum_{k=1}^{\infty} |z^{\epsilon_k}|^2 (2k)^q \right)^{\frac{1}{2}}, \end{aligned}$$

and thus

$$\begin{aligned}
|\mathbf{I}(W_m(t))(z) - \mathbf{I}(W_m(s))(z)| &\leq \left(\sum_{k=1}^{\infty} \left| \left\{ (T_m \tilde{h}_k)(t) - (T_m \tilde{h}_k)(s) \right\} \right|^2 (2k)^{-q} \right)^{\frac{1}{2}} \times \\
&\quad \times \left(\sum_{\alpha \in \ell} |z^\alpha|^2 (2\mathbb{N})^{\alpha q} \right)^{\frac{1}{2}} \\
&\leq |t - s| \left(\sum_{k=1}^{\infty} \left\{ \tilde{C}_1 k^{\frac{11}{12}} + \tilde{C}_2 \right\}^2 (2k)^{-q} \right)^{\frac{1}{2}} \times \\
&\quad \times \left(\sum_{\alpha \in \ell} |z|^\alpha (2\mathbb{N})^{\alpha q} \right)^{\frac{1}{2}},
\end{aligned}$$

where we have used (3.11) to go from the first inequality to the second.

The first sum converges for $q \geq 4$ and the second converges since $z \in K_4(\delta)$, so we conclude that $\mathbf{I}(W_m(t))(z)$ is continuous in t for every $z \in K_4(\delta)$. \square

We now recall the following result of [22], called *the differentiation of S_{-1} processes*.

Proposition 7.6. [22, Lemma 2.8.4 p. 77] *Suppose $\{X(t, \omega)\}$ and $\{F(t, \omega)\}$ are S_{-1} -valued processes such that*

$$\frac{d(\mathbf{I}(X)(t))(z)}{dt} = (\mathbf{I}(F)(t))(z)$$

for each $t \in (a, b)$, $z \in K_q(\delta)$ and that $(\mathbf{I}(F)(t))(z)$ is bounded for $(t, z) \in (a, b) \times K_q(\delta)$, and is a continuous function of t for every $z \in K_q(\delta)$. Then $X(t, \omega)$ is a differentiable process and

$$\frac{dX(t, \omega)}{dt} = F(t, \omega)$$

for all $t \in (a, b)$.

In view of this proposition, the first step toward showing that W_m is the derivative of B_m is to show that this fact holds for the Hermite transforms. This is done in the following lemma.

Lemma 7.7. *Let $t \in \mathbb{R}$ and $z \in K_4(\delta)$. Then*

$$\frac{d\mathbf{I}(B_m(t))(z)}{dt} = \mathbf{I}(W_m(t))(z).$$

Proof: Let $h \in \mathbb{R}$. Then

$$\begin{aligned}
& \left| \frac{\mathbf{I}(B_m(t+h))(z) - \mathbf{I}(B_m(t))(z)}{h} - \mathbf{I}(W_m(t))(z) \right| \\
&= \frac{1}{|h|} \left| \sum_{k=1}^{\infty} \int_t^{t+h} \left((T_m \tilde{h}_k)(s) - (T_m \tilde{h}_k)(t) \right) ds z_k \right| \\
&= \frac{1}{|h|} \left| \sum_{k=1}^{\infty} \int_t^{t+h} \left((T_m \tilde{h}_k)(s) - (T_m \tilde{h}_k)(t) \right) ds (2k)^{-\frac{q}{2}} (2k)^{\frac{q}{2}} z_k \right| \\
&\leq \frac{1}{|h|} \left(\sum_{k=1}^{\infty} \left| \int_t^{t+h} \left((T_m \tilde{h}_k)(s) - (T_m \tilde{h}_k)(t) \right) ds \right|^2 (2k)^{-q} \right)^{\frac{1}{2}} \cdot \\
&\quad \left(\sum_{k=1}^{\infty} (2k)^q |z^{\epsilon_k}|^2 \right)^{\frac{1}{2}} \\
&\leq \frac{1}{|h|} \left(\sum_{k=1}^{\infty} \int_t^{t+h} \left| (T_m \tilde{h}_k)(s) - (T_m \tilde{h}_k)(t) \right|^2 ds (2k)^{-q} \right)^{\frac{1}{2}} \cdot \\
&\quad \left(\sum_{\alpha \in \ell} (2\mathbb{N})^{q\alpha} |z^\alpha|^2 \right)^{\frac{1}{2}},
\end{aligned}$$

and therefore

$$\begin{aligned}
& \left| \frac{\mathbf{I}(B_m(t+h))(z) - \mathbf{I}(B_m(t))(z)}{h} - \mathbf{I}(W_m(t))(z) \right| \\
&\leq \frac{1}{|h|} \left(\sum_{k=1}^{\infty} \int_t^{t+h} |t-s|^2 ds \left\{ \tilde{C}_1 k^{\frac{11}{12}} + \tilde{C}_2 \right\}^2 (2k)^{-q} \right)^{\frac{1}{2}} \cdot \\
&\quad \left(\sum_{\alpha \in \ell} (2\mathbb{N})^{q\alpha} |z^\alpha|^2 \right)^{\frac{1}{2}} \\
&\leq \frac{|h|^{\frac{3}{2}}}{\sqrt{3}|h|} \left(\sum_{k=1}^{\infty} \left\{ \tilde{C}_1 k^{\frac{11}{12}} + \tilde{C}_2 \right\}^2 (2k)^{-q} \right)^{\frac{1}{2}} \cdot \\
&\quad \left(\sum_{\alpha \in \ell} (2\mathbb{N})^{q\alpha} |z^\alpha|^2 \right)^{\frac{1}{2}} \longrightarrow_{|h| \rightarrow 0} 0.
\end{aligned}$$

□

We are now ready for the main result of this section:

Theorem 7.8. *It holds that*

$$\frac{dB_m(t)}{dt} = W_m(t)$$

in the sense that

$$\frac{d(\mathbf{I}(B_m(t))(z))}{dt} = \mathbf{I}(W_m(t))(z)$$

for all $t \in \mathbb{R}$, point-wise boundedly.

Proof: Taking in Proposition 7.6, $X(t, \omega) = B_m(t)$, $F(t, \omega) = W_m(t)$ we have from Lemma 7.7

$$\frac{d(\mathbf{I}(B_m(t))(z))}{dt} = \mathbf{I}(W_m(t))(z)$$

for all $(t, z) \in \mathbb{R} \times K_4(\delta)$, and by Proposition 7.4, $\mathbf{I}(W_m(t))(z)$ is a bounded function for all $(t, z) \in \mathbb{R} \times K_2(\delta)$, then for all $(t, z) \in \mathbb{R} \times K_4(\delta)$ the pair $(\mathbf{I}(B_m(t))(z), \mathbf{I}(W_m(t))(z))$ satisfied the condition of Proposition 7.6 then we can conclude that $B_m(t)$ is a differentiable in S_{-1} process which completes the proof. \square

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